MATHEMATICS

NOT LOCALLY COMPACT MONOTHETIC GROUPS. II.

BY

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§ 6. THE COMPUTATION OF THE NORM IN DS-GROUPS

In this section we want to obtain as detailed as possible information on the numerical value of the norm $\|\star\|$ in a DS-group. We will try to obtain information on the set of places on which y has to be zero, so that it be minimizing for z.

42. Lemma. Let $z \in \mathbf{Z}^+$ be given. Suppose $Q(j, z, y') \neq Q(j, z, y)$ and

$$|P_{j-1}y| + |P_{j-1}(z+y)| + p_j < |P_{j-1}y'| + |P_{j-1}(z+y')|,$$

then y' is not minimizing.

Proof. Suppose y' minimizing and put $Q(j, z, y) = r$ and let $y'' = P^j y''$ be minimizing for $P^j z + r n_j$. Put $y''' = P_{j-1} y + y''$.

Then

$$\begin{aligned}
&\|z\| \leqq |z + y'''| + |y'''| = \\
&= |P_{j-1}(z+y)| + |P^j z + r n_j + y''| + |P_{j-1} y| + |y''| = \\
&= |P_{j-1}(z+y)| + \|P^j z + r n_j\| + |P_{j-1} y| \leqq \\
&\leqq |P_{j-1}(z+y)| + |P_{j-1} y| + \|P^j z + (1-r) n_j\| + p_j = (\text{by } 29.) \\
&= |P_{j-1}(z+y)| + |P_{j-1} y| + |P^j z + (1-r) n_j + P^j y'| + |P^j y'| + p_j < \\
&< |P_{j-1}(z+y')| + |P_{j-1} y'| + |P^j z + (1-r) n_j + P^j y'| + |P^j y'| = \\
&= |z + y'| + |y'| = \|z\|,
\end{aligned}$$

which is a contradiction.

43. Lemma. Let $z \in \mathbf{Z}^+$ be given. Suppose $Q(j, z, y) = Q(j, z, y')$ and

$$|P_{j-1}y| + |P_{j-1}(z+y)| < |P_{j-1}y'| + |P_{j-1}(z+y')|.$$

Then y' is not minimizing.

Proof. Obvious.

To illustrate the remainder of the section, we give an easy example: $n_i = 2^i$, $p_i > p_{i+1}$, for all i. In this case, if $z = \sum a_i n_i$ is positive reduced, a_i equals 0 or 1 for all i.

Let z be given and let y be minimizing for z. Then

44. Lemma. One of the following statements holds, for $t \in \mathbf{N}$

(i) $z = \ldots a_{t-1}, 1, 1, \ldots$; $Q(t, y, z) = 0$ and, by 42., $y = \ldots b_{t-1}, 1, 0, \ldots$.

(ii) $z = \ldots a_{t-1}, a_t, a_{t+1}, \ldots$; $Q(t, y, z) = 0$ and not $a_t = a_{t+1} = 1$ and $y = \ldots 0, 0, 0, \ldots$.

(iii) $z = \ldots a_{t-1}, 1, \ldots$; $Q(t, y, z) = 1$ and, by 11., $y = \ldots b_{t-1}, 0, \ldots$.

(iv) $z = \ldots a_{t-1}, 0, 1, \ldots$ and $Q(t, y, z) = 1$ and, by 42., $y = \ldots b_{t-1}, 1, 0, \ldots$.

(v) $z = \ldots a_{t-1}, 0, 0, \ldots$, and, by 31., $y = \ldots b_{t-1}, 0, 0, \ldots$.

From this can easily be seen what the minimizing y is for a given $z = \sum a_i n_i$. It is determined by the set $K(z) = \{t \in \mathbf{N} | Q(t, y, z) = 1\}$, or even by the set $C(z) = \{t \in \mathbf{N} | Q(t, y, z) \neq Q(t+1, y, z)\}$, if we define $Q(0, y, z) = 0$ for all y, z. We see that the smallest element t_1 of $C(z)$ is the smallest t such that $z = \ldots a_{t-1}, 1, 1, \ldots$.

If $C(z)$ is non-empty, the second element is the smallest $t > t_1$ such that $z = \ldots a_{t-1}, 0, 0, \ldots$. Generally, t_{2n+1} is the smallest $t > t_{2n}$ such that $z = \ldots a_{t-1}, 1, 1, \ldots$ and t_{2n+2} is the smallest $t > t_{2n+1}$ such that $z = \ldots \ldots a_{t-1}, 0, 0, \ldots$. $C(z)$ always has an even number of elements.

The reader may, if he wishes, skip the rest of this section, as it will not be used for subsequent results.

Throughout the rest of this section, $z = \sum a_i n_i$ will denote a fixed element of $\mathbf{Z}^+$.

45. Lemma. Let y be minimizing for z. Suppose

$$(1-2q)p_j + \textstyle\sum_{i=j}^{m} (k_i - 2a_i - 1)p_i + (1-2r)p_{m+1} < 0.$$

Then $Q(j, z, y) = q$ and $Q(m+2, z, y) = r$ and $Q(i, z, y) = 0$ for $j+1 \leqq i \leqq m+1$, does not hold.

Proof. Suppose the contrary. It follows that, putting $y = \sum b_i n_i$ and $y + z = \sum c_i n_i$, $b_i = 0$ for $j \leqq i \leqq m$, $b_{m+1} = (k_{m+1} - a_{m+1})r$ and $c_j = a_j + q$, $c_i = a_i$ for $j+1 \leqq i \leqq m$, $c_{m+1} = a_{m+1}(1-r)$. Put $y' = \sum b_i' n_i$, $z + y' = \sum c_i' n_i$, such that $b_i' = b_i$ for $i < j$ and $i > m+1$, and $b_j' = k_j - a_j - q$, $b_i' = k_i - a_i - 1$, for $j+1 \leqq i \leqq m$, $b'_{m+1} = (k_{m+1} - a_{m+1} - 1)r$ and $c_i' = 0$ for $j \leqq i \leqq m$, $c'_{m+1} = (a_{m+1}+1)(1-r)$, $c_i' = c_i$ for $i < j$, and $i > m+1$.

We remark that the decomposition $z + y' = \sum c_i' n_i$ is possibly not reduced, so $z + y' \leqq \sum c_i' p_i$. Consequently

$$\begin{aligned} 0 &\leqq |z+y'| + |y'| - |z+y| - |y| \leqq \textstyle\sum (c_i' + b_i' - c_i - b_i)p_i = \\ &= (1-2q)p_j + \textstyle\sum_{i=j}^{m} (k_i - 2a_i - 1)p_i + (1-2r)p_{m+1} < 0, \end{aligned}$$

which is a contradiction. We have proved the lemma.

We introduce now some notations.

46. Definition. For any pair $(j, m) \in \mathbf{N} \times \mathbf{N}$, $j \leqq m$, let

$$s(j, m) = \sum_{i=j}^{m} (k_i - a_i - 1)p_i;$$
$$s_+^+(j, m) = p_j + s(j, m) + p_{m+1};$$
$$s_-^+(j, m) = -p_j + s(j, m) + p_{m+1};$$
$$s_+^-(j, m) = p_j + s(j, m) - p_{m+1};$$
$$s_-^-(j, m) = -p_j + s(j, m) - p_{m+1}.$$

We now reformulate 45.

47. Lemma. Let y be minimizing for z. Then each of the conditions (i), (ii), (iii), (iv) implies that the corresponding statement (i′), (ii′), (iii′), (iv′) does not hold:

(i) $s_+^+(j, m) < 0$: (i′) $Q(i, z, y) = 0$ for $j+1 \leqq i \leqq m+1$

(ii) $s_-^+(j, m) < 0$: (ii′) $Q(i, z, y) = 0$ for $j+1 \leqq i \leqq m+1$ and $Q(j, z, y) = 1$.

(iii) $s_+^-(j, m) < 0$: (iii′) $Q(i, z, y) = 0$ for $j+1 \leqq i \leqq m+1$ and $Q(m+2, z, y) = 1$.

(iv) $s_-^-(j, m) < 0$: (iv′) $Q(i, z, y) = 0$ for $j+1 \leqq i \leqq m+1$ and $Q(j, z, y) = Q(m+2, z, y) = 1$.

Proof. Obvious from 45. Remark that $s_-^-(j, m) < s_-^+(j, m) < s_+^+(j, m)$ and that $s_-^-(j, m) < s_+^-(j, m) < s_+^+(j, m)$.

48. Definition. A pair (j, m) will be called of

Type 0 when $s_-^-(t, l) < 0$ for all $j \leqq t \leqq l \leqq m$,
Type I when $s_+^+(j, m) < 0$,
Type II when $s_-^+(t, m) < 0$ for all $t > j$,
Type II* when $s_+^-(j, l) < 0$ for all $l < m$,
Type II_0 when (j, m) is of type 0 and $(j-1, m)$ of type II,
Type II_0* when (j, m) is of type 0, and $(j, m+1)$ of type II*,
Type III when (j, m) is of types 0, I, II, II*.

49. Definition. (j', m') is *contained* in (j, m) when $j \leqq j' \leqq m' \leqq m$. We will say $i \in (j, m)$ if and only if (i, i) is contained in (j, m). The composition of two pairs (j, m) and (j', m') is defined to be (j, m') if and only if $j' = m+1$; (j, m') is then said to be the composition of (j, m) followed by (j', m') or of (j', m') preceded by (j, m).

50. Lemma. Let y be minimizing for z. Suppose (j, m) is a pair of type III. Then $Q(i+1, y, z) = 1$ if $i \in (j, m)$.

Proof. Because (j, m) is of type 0, for some $i \in (j, m)$, $Q(i+1, y, z) = 1$. Now suppose $Q(j+1, y, z) = 0$. For some l such that $j+1 \leqq l \leqq m+1$, $Q(i, y, z) = 0$ for $j+1 \leqq i < l$ and $Q(l, y, z) = 1$. This contradicts the fact that $s_+^-(j, l-2) < 0$ for $l-2 \leqq m-1$ and hence contradicts the fact that (j, m) is of type II*. So $Q(j+1, y, z) = 1$.

Suppose $Q(m+1, y, z)=0$. For some t such that $j+1 \leqq t \leqq m$, $Q(t, y, z)=1$ and $Q(i, y, z)=0$ for $t<i \leqq m+1$. This contradicts that (j, m) is of type II. so $Q(m+1, y, z)=1$. As (j, m) is of type 0, the existence of any pair (j', m'), contained in (j, m) such that $Q(j', y, z)=Q(m'+2, y, z)=1$ and $Q(i+1, y, z)=0$ for all $i \in (j', m')$ is impossible, whence the result.

III-type pairs may be found by the following

51. Lemma. Let (j, m) be a pair of type I (relative to z). Suppose that no pair (j', m') contained in (j, m) is of type I. Then (j, m) is of type III.

Proof. $s_+^+(j, m)<0$. For any (t, l) contained in (j, m), $s_+^+(t, l) \geqq 0$, hence, since

$s_+^+(j, m)=s_+^-(j, t)+s_+^+(t+1, m)=s_+^+(j, l-1)+s_-^+(l, m)=$
$=s_+^+(j, t-1)+s_-^-(t, l)+s_+^-(l+1, m)$,
$s_+^-(j, t)$, $s_-^+(l, m)$ and $s_-^-(t, l)$ are all strictly negative.

52. Lemma. The composition of a III-type pair, preceded by a $\mathrm{II}_0{}^*$-type pair or followed by a II_0-type pair is a III-type pair; because a III-type pair is both of type II_0 and $\mathrm{II}_0{}^*$, the composition of two III-type pairs is again a III-type pair.

Proof. We prove the first statement, and leave the others to the reader. So, let (j, m) be of type $\mathrm{II}_0{}^*$ and $(m+1, m')$ of type III. For t, l, such that $t<m<l$, $s_-^-(t, l)=s_-^-(t, m)+s_+^-(m+1, l)<0$. So (j', m) is of type 0. Secondly, $s_+^+(j, m')=s_+^+(j, m)+s_+^+(m+1, m')<0$. So (j, m') is of type I. Now for $j<t \leqq m$: $s_-^+(t, m')=s_-^-(t, m)+s_+^+(m+1, m')<0$ and for $m+1<l<m'$: $s_+^-(j, l)=s_+^-(j, m)+s_+^-(m+1, l)<0$. So (j, m) is of type II and II*.

53. Lemma. Let (j, m) be a III-type pair, then for any t such that $j \leqq t \leqq m$, (j, t) is of type $\mathrm{II}_0{}^*$ and (t, m) is of type II_0.

Proof. Left to reader.

54. Lemma. Let (j, m) and (j', m') be III-type pairs and suppose $(m+1, j'-1)$ is of type 0. Then (j, m') is of type III.

Proof. $s_+^+(j, m')=s_+^+(j, m)+s_-^-(m+1, j'-1)+s_+^+(j', m')<0$ hence (j, m') is of type I. $s_-^-(t, l)=s_-^+(t, m)+s_-^-(m+1, j'-1)+s_+^-(j', l) \leqq 0$, if $t \leqq m \leqq j'<l$; $s_-^-(t, l)=s_-^+(t, m)+s_-^+(m+1, l)<0$ if $t \leqq m \leqq l \leqq j'$ and $s_-^-(t, l)=s_-^-(t, j'-1)+s_+^-(j', l)$ if $m \leqq t \leqq l \leqq j'$, hence (j, m') is of type 0. For $j<t<m$: $s_-^+(t, m')=s_-^+(t, m)+s_-^-(m+1, j'-1)+s_+^+(j', m')<0$ and for $m+1 \leqq t<j'-1$: $s_-^+(t, m')=s_-^-(t, j'-1)+s_+^+(j', m')<0$. Hence (j, m') of type II. Now for $j' \leqq l \leqq m'$: $s_+^-(j, l)=s_+^+(j, m)+s_-^-(m+1, j'-1)+$ $+s_+^-(j', l)<0$ and for $m+1 \leqq l \leqq j'-1$: $s_+^-(j, l)=s_+^+(j, m)+s_-^-(m+1, l)<0$. We are ready.

55. Definition. A pair (j, m) is called an *F-pair*, if there exists a $t \in (j, m)$ such that (j, t) is of type III, and $a_i = k_i - 1$ for $i \in (t+1, m)$.

Remark. When $p_m k_m - p_{m+1}$, then (j, m) may be an F-pair, but not a III-type pair.

Evidently, we have though:

56. Lemma. Let $K_\pi(z)$ be the set of all i, such that there exists an F-pair (j, m), such that $i \in (j, m)$. Then $Q(i+1, y, z) = 1$, if $i \in K_\pi(z)$ and hence $b_i = k_i - a_i - Q(i, y, z)$.

Proof. Evident.

We will now prove that the minimizing y may be chosen such that $Q(i+1, y, z) = 0$, for i in the complement of $K_\pi(z)$.

57. Lemma. Let $y = \sum b_i n_i \in \mathbf{Z}^+$ be minimizing for z. Suppose that for a certain pair (j, m), $(1-2q)p_j + \sum_{i=j}^{m} (k_i - 2a_i - 1)p_i + (1-2r)p_{m+1} \geq 0$, and $a_j + q \neq k_j$. Then, if $Q(j, z, y) = q$, $Q(m+2, y, z) = r$ and $Q(i, z, y) = 1$, for $i \in (j, m)$, $y' = P_{j-1}y + r(k_{m+1} - a_{m+1}) + P^{m+2}y$ is minimizing for z.

Proof. $b_j = k_j - a_j - q$; $b_i = k_i - a_i - 1$ for $j+1 \leqq i \leqq m+1$ and $b_{m+1} = (k_{m+1} - a_{m+1} - 1)r$. So, if $y + z = \sum c_i n_i$, $c_i = 0$ for $j \leqq i \leqq m$; and $c_{m+1} = (1 + a_{m+1})(1-r)$. Then, putting $y' + z = \sum c_i' n_i$, in which $c_j' = q + a_j < k_j$ and $c_i = a_i$ for $j+1 \leqq i \leqq m$ and $c'_{m+1} = a_m(1-r)$. We have

$$0 \geq |y+z| + |y| - |y'+z| - |y'| = \sum_{i=j}^{m+1} (b_i + c_i - b_i' - c_i')p_i =$$
$$= (1-2q)p_j + \sum_{i=j}^{m} (k_i - 2a_i - 1)p_i + (1-2r)p_{m+1} \geq 0.$$

Hence equality holds throughout, so y' is minimizing.

58. Lemma. Let (j, m) be such that for every pair (t, l) contained in (j, m), (j, t) is not of type II_0, (l, m) is not of type II_0^* and (t, l) is not of type I; let moreover (j, m) not be of type 0, then: $s_-^+(j, t) \geq 0$, $s_+^-(l, m) \geq 0$, $s_+^+(t, l) \geqq 0$, $s_-^-(j, m) \geqq 0$, for all (t, l) contained in (j, m).

Proof. We prove first that $s_-^+(j, t) \geqq 0$ for all $t \leqq m$. (j, j) is not of type II_0, hence $s_-^+(j, j) \geqq 0$ or $s_-^-(j, j) \geqq 0$. As $s_-^+(j, j) \geqq s_-^-(j, j)$, we conclude that $s_-^+(j, j) \geqq 0$. Let $s_-^+(j, i) \geqq 0$ for $j \leqq i \leqq t$. $(j, t+1)$ is not of type II_0, hence one of the following statements holds:

(*) $s_-^-(t', t'') \geqq 0$ for (t', t'') contained in (j, t).

(**) $s_-^+(t', t+1) \geqq 0$ for $t' \in (j, t+1)$.

In case (*), $s_-^+(j, t+1) \geqq s_-^+(j, t'-1) + s_-^-(t', t'') + s_+^+(t''+1, t+1) \geqq 0$, if the first summand is set to be 0 if $t' = j$.

In case (**), $s_-^+(j, t+1) = s_-^+(j, t'-1) + s_-^+(t', t+1) \geqq 0$, if the first summand is set to be 0 if $t' = j$. So $s_-^+(j, t+1) \geqq 0$.

Similarly one proves $s_+^-(l, m) \geqq 0$ for all $l \leqq m$. This is left to the reader.

Now, if (j, m) is not of type 0, then, for some (t, l) contained in (j, m), $s_-^-(t, l) \geqq 0$. But then $0 \leqq s_-^+(j, t-1) + s_-^-(t, l) + s_+^-(l+1, m) = s_-^-(j, m)$. (Again, a summand can be put equal 0, if it does not make sense). We have proved the lemma.

59. Lemma. Let (j, m) be such that $j-1 \in K_\pi(z)$, $m+1 \in K_\pi(z)$ and if $i \in (j, m)$ then $i \notin K_\pi(z)$. Then there exists a minimizing y such that $Q(i+1, y, z) = 0$ for $i \in (j, m)$.

Proof. Evidently, because of 51.–54., (j, m) satisfies the conditions of 58. Furthermore 57. applies, for y minimizing with $Q(j, y, z) = Q(m+2, y, z) = 1$. So 57. implies that there is a minimizing y such that $Q(i, y, z) = 1$, for $j < i < t$. Applying 57. in the case $q = 1$, $r = 0$, shows that we may as well suppose $Q(i, y, z) = 0$ for $j+1 \leqq i \leqq t$. Likewise we apply 57. in the case $q = 0$, $r = 1$, to obtain a y such that $Q(i, y, z) = 0$ for $j+1 \leqq i \leqq t$ and $l \leqq i \leqq m+1$. A finite number of applications of lemma 57. in the case $q = r = 0$ gives the lemma.

60. Lemma. There exists a minimizing y such that: $Q(i+1, y, z) = 1$ if and only $i \in K_\pi(z)$.

Proof. Clear from the above.

§ 7. Topology and metric of different DS-groups compared with each other

Suppose we are give $\pi = \{\{n_i\}, \{p_i\}\}$ and $\pi' = \{\{n_i'\}, \{p_i'\}\}$. Is the π-topology on $\mathbf{Z}$ coarser, finer or equal to the π'-topology on $\mathbf{Z}$? Are they comparable at all? We will give a necessary condition for $\vartheta = \{n_i\}$ and $\vartheta' = \{n_i'\}$ for equality and in case $\vartheta = \vartheta'$ we will give necessary and sufficient conditions on $\{p_i\}$, $\{p_i'\}$, for any of the possibilities. One will note that this does not settle everything, in particular not the case that $\vartheta \neq \vartheta'$, $G_\vartheta = G_{\vartheta'}$. Furthermore, this does not prove anything about the question: are all not locally compact complete DS-groups G homeomorphic or isomorphic as topological groups? Connected with this is the unsolved problem, to determine which elements of $G_\pi \backslash \mathbf{Z}$ generate a dense subgroup of G_π.

In the following we will write $\pi = \{\vartheta, \varrho\}$, in which $\vartheta = \{n_i\}$, and $\varrho = \{p_i\}$. It is understood, as before, that $n_i p_{i+1} \leqq n_{i+1} p_i$. The DS-norm defined by π is indicated by $\|\star\|_\pi$. Instead of $\langle \mathbf{Z}, \|\star\|_\pi \rangle$ we will write $\langle \mathbf{Z}, \pi \rangle$. $\langle \mathbf{Z}, \vartheta \rangle$ will denote $\mathbf{Z}$ with the ϑ-topology. The semigroupnorm on $\mathbf{Z}^+$, defined by π, will henceforth be denoted by $|\star|_\pi$. For $\pi = \{\varrho, \vartheta\}$ we will consider G_π as a subgroup of G_ϑ, but with its own (finer) topology. This is allowed because of 34. We will denote by S_π the set

$$\{ \textstyle\sum_{i=0}^\infty a_i n_i \in G_\pi | \sum_{i=0}^\infty a_i p_i < \infty \text{ and } a_i \geqq 0 \text{ for all } i\},$$

in other words, $S_\pi \subset G_\pi \subset G_\vartheta$ is the set of all $z \in G_\pi$ such that $|z|_\pi$ is defined.

61. Lemma. Let $z \in \mathbf{Z}^+$ and $\pi = \{\vartheta, \varrho\}$. Then, if G_π is not discrete, the closure of the subgroups generated by z in G_π and G_ϑ are of the same index in G_π and G_ϑ, namely $\lim_{i \in \mathrm{N}} GCD(z, n_i) = z_\vartheta$.

Proof. Let j be the index at which $GCD(z, n_i)$ stabilizes, that means $GCD(z, n_j) = GCD(z, n_i)$ for all $i \in \mathrm{N}$, $i > j$. Then for every $i \geqq j$ there exist integers p and q such that

$$zp = z + n_i q, \text{ and } q < z.$$

This implies that z_ϑ can be approximated arbitrarily close by multiples of z, when there exist arbitrarily small p_i, that is, when G_π is not discrete. The subgroup $z_\vartheta \mathbf{Z}$ is closed in the ϑ-topology, because $z_\vartheta | n_j$, and $n_j \mathbf{Z}$ is closed in $\mathbf{Z}$, and of finite index, namely n_j / z_ϑ in $z_\vartheta \mathbf{Z}$.

62. Lemma. Let $\pi = \{\vartheta, \varrho\}$ and $\pi' = \{\vartheta', \varrho'\}$. For $1_{\mathbf{Z}} : \langle \mathbf{Z}, \pi \rangle \to \to \langle \mathbf{Z}, \pi' \rangle$ to be continuous it is necessary that $1_{\mathbf{Z}} : \langle \mathbf{Z}, \vartheta \rangle \to \langle \mathbf{Z}, \vartheta' \rangle$ is continuous.

Proof. For $1_{\mathbf{Z}}$ to be continuous, it is necessary that the set of closed subgroups relative to the π'-topology on $\mathbf{Z}$ is a subset of the set of closed subgroups relative to the π-topology. Equivalently, it is necessary that the set of closed subgroups relative the ϑ'-topology on $\mathbf{Z}$ is a subset of the set of closed subgroups relative the ϑ-topology. Because the set of closed subgroups form a neighborhoodbasis at 0 for the ϑ- and ϑ'-topology, we have proved our statement.

Comment. If $\vartheta \neq \vartheta'$, but $G_\vartheta = G_{\vartheta'}$ it is difficult to say anything. Suppose, for example, that $\pi = \{\{n_i\}, \{p_i\}\}$, with $n_i = 2^i$ for all $i \in \mathrm{N}$, and $p_i = 2^{-i}$ for all $i \in \mathrm{N}$. Choose $\pi' = \{\{n_i'\}, \{p_i'\}\}$, with $n_0' = p_0' = 1$ and $n_i' = 2^{2^i}$ for $i > 0$, and $p_i' = (n_i')^{-1}$ for $i > 0$. Then $G_\pi = G_\vartheta = G_{\vartheta'} =$ the 2-adic integers but, for the π'-norm, $\|2^{2^{i-1} \cdot 3}\| = \frac{1}{2}$ holds for $i > 0$, so $G_{\pi'}$ is properly contained in $G_{\vartheta'} = G_\vartheta$.

The case $\vartheta = \vartheta'$ is easier. We will assume throughout the rest of this section, that ϑ is fixed. As notational convention we let $\pi' = \{\vartheta, \varrho'\}$, $\pi'' = \{\vartheta, \varrho''\}$, $\varrho' = \{p_i'\}$ and $\varrho'' = \{p_i'\}$ etc.

63. Lemma. $1_{\mathbf{Z}} : \langle \mathbf{Z}, \pi \rangle \to \langle \mathbf{Z}, \pi' \rangle$ is continuous if and only if $S_\pi \subset S_{\pi'}$, if and only if $G_{\pi'} \supset G_\pi$, as subgroups of G_ϑ.

Proof. (i) Suppose $S_{\pi'} \not\supset S_\pi$. So we have a $z \in G_\vartheta$ for which $|z|_\pi$ is defined, but $|z|_{\pi'}$ not. Put $z = \sum_{i=0}^\infty a_i n_i$, then $\sum_{i=0}^\infty a_i p_i < \infty$ and $\sum_{i=0}^\infty a_i p_i' = \infty$. Define b_i, $0 \leqq b_i < k_i$ as follows. If $k_i = 2$, then $b_i = a_i$. If $k_i > 2$, then $b_i = \min(a_i, [(k_i - 1)/2])$. We conclude that $\sum_{i=0}^\infty b_i p_i < \infty$, and $\sum_{i=0}^\infty b_i p_i' = \infty$. From this we have $\sum_{i=0}^\infty b_{2i} p_{2i}' = \infty$ or $\sum_{i=0}^\infty b_{2i+1} p_{2i+1}' = \infty$. As in 23., we find a sequence $\{z_M | M \in \mathrm{N}\}$ which is unbounded with respect to $\|\star\|_{\pi'}$, but which is a Cauchy sequence with respect to $\|\star\|_\pi$. Hence $1_{\mathbf{Z}} : \langle \mathbf{Z}, \pi \rangle \to \to \langle \mathbf{Z}, \pi' \rangle$ is not continuous.

(ii) Suppose $1_{\mathbf{Z}}: \langle \mathbf{Z}, \pi \rangle \to \langle \mathbf{Z}, \pi' \rangle$ is not continuous. Then we have a sequence $\{z_i \in \mathbf{Z}^+ | i \in \mathbf{N}\}$ such that $\|z_i\|_\pi$ converges to 0, and $\|z_i\|_{\pi'}$ is bounded away from 0, say $0 < B < \|z_i\|_{\pi'}$. If $\langle \mathbf{Z}, \pi' \rangle$ is discrete, clearly $G_{\pi'} \not\supset G_\pi$, so we may assume $\liminf_{i \in \mathbf{N}} p_i' = 0$. Extract a subsequence from $\{z_i\}$, say $\{x_i\}$ with the following properties:

(a) $\sum \|x_i\|_\pi$ converges.
(b) for every i there exists a $j(i)$ such that $x_i < n_j$, $p'_{j+1} < B/4$, $n_{j+2} | x_{i+1}$ for $j = j(i)$.

Then $\sum_{i=s}^{t} \|x_i\|_{\pi'} \geqq \sum_{i=s}^{t} (\|x_i\|_{\pi'} - 2p'_{j(i)+1}) > (t-s)B/2$ (we have applied 28.).
On the other hand, $\| \sum_{i=s}^{t} x_i \|_\pi \leqq \sum_{i=s}^{t} \|x_i\|_\pi$ and we conclude that $\sum_{i=0}^{\infty} x_i \in G_\pi \backslash G_{\pi'}$. So $G_{\pi'} \not\supset G_\pi$.
(iii) Suppose $G_{\pi'} \not\supset G_\pi$ and $S_{\pi'} \supset S_\pi$. $G_{\pi'} = S_{\pi'} - S_{\pi'} \supset S_\pi - S_\pi = G_\pi$, so this is a contradiction. $G_{\pi'} \not\supset G_\pi$ implies $S_{\pi'} \not\supset S_\pi$.

64. Definition. For any $M \in \mathbf{R}^+$, $M > 0$, and π define

$$S_\pi(M) = \{ \textstyle\sum_{i=0}^{\infty} a_i n_i \in S_\pi \mid a_i \neq 0 \text{ implies } p_i < M\}$$

and

$$S^\pi(M) = \{ \textstyle\sum_{i=0}^{\infty} a_i n_i \in S_\theta \mid a_i \neq 0 \text{ implies } p_i < M\}.$$

Clearly $S^\pi(M) \cap S_\pi = S_\pi(M)$.

65. Lemma. $S_{\pi'} \supset S_\pi$ if and only if for every $M' > 0$ there exists an $M > 0$ such that $S_{\pi'}(M') \supset S_\pi(M)$.

Proof. To begin with, we remark that $S_\pi(M) + \mathbf{Z}^+ = S_\pi$, so the if-part is trivial. Suppose $\langle \mathbf{Z}, \pi' \rangle$ is non-discrete and $S_{\pi'} \supset S_\pi$. Let M' be given. There exists an M such that $p_i < M$ implies $p_i' < M'$, for suppose not, then there would be a set $V = \{i(k) | k \in \mathbf{N}\} \subset \mathbf{N}$ such that $p_i < 2^{-k}$ and $p_i' \geqq M'$ for $i = i(k) \in V$. Hence $\sum_{i \in V} n_i \in S_\pi \backslash S_{\pi'}$.
So for some M, $S^{\pi'}(M') \supset S^\pi(M)$; together with $S_{\pi'} \supset S_\pi$, this implies $S_{\pi'}(M') \supset S_\pi(M)$. Now suppose $\langle \mathbf{Z}, \pi' \rangle$ is discrete, then $\langle \mathbf{Z}, \pi \rangle$ is discrete too, so we can take $M = \frac{1}{2} \liminf \{p_i\} \neq 0$ and $S_\pi(M) = \{0\}$. We have proved the "only if"-part.

66. Definition. For π given, $V_\pi(M) = \{i \in \mathbf{N} | \, p_i < M\}$.

67. Lemma. Let π, π' be given. Then for every M', there exists an $M > 0$ such that $S_{\pi'}(M') \supset S_\pi(M)$ if and only if there exists an $M > 0$ such that $V_{\pi'}(M') \supset V_\pi(M)$ and a set of real numbers

$$\{E\} \cup \{r_i | i \in V(M)\} \subset \mathbf{R}^+ \text{ such that } p_i'/r_i < E \text{ for } i \in V(M) \text{ and}$$
$$\textstyle\sum \{k_i |p_i - r_i| \mid i \in V(M)\} \text{ exists.}$$

Proof. (i) "If": $x \in S_\pi(M)$ means that $x = \sum_{i \in V(M)} a_i n_i$ such that $\sum_{i \in V(M)} a_i p_i < \infty$. Hence $x \in S^{\pi'}(M')$ and $\sum_{i \in V(M)} (a_i(r_i - p_i) + a_i p_i) < \infty$, hence $|x|_{\pi'} = \sum_{i \in V(M)} a_i p_i' < E \sum_{i \in V(M)} a_i r_i < \infty$. So $x \in S_{\pi'} \cap S^{\pi'}(M')$.

(ii) "Only if". Evidently $S_{\pi'}(M') \supset S_\pi(M)$ implies $V_{\pi'}(M') \supset V_\pi(M)$.

Suppose that $\{p_i'/p_i | i \in V(M)\}$ is unbounded. Define for any real number $E > 0$, the set $F_E = \{i \in V(M) | p_i'/p_i \geqq E\}$, and put $F_{2^k} = F^k$, for k a positive integer. All F^k are infinite. Now $S_{\pi'}(M') \supset S_\pi(M)$ implies that, for some E, $\sum_{i \in F_E} k_i p_i' < \infty$.

For suppose not. Choose finite sets D_t, $t \in \mathbf{N}$ and a_i for $i \in D = \bigcup_{t \in \mathbf{N}} D_t$ as follows: $D_0 = \emptyset$, $D_t \subset F^t$ is disjoint from $\bigcup_{0 \leq j < t} D_j$ and $1 < \sum_{i \in D_t} p_i' a_i \leqq \leqq M' + 1$, for some choice of the a_i. Then, for $z = \sum_{i \in D} a_i n_i$, we see that $|z|_{\pi'} = \sum_{i \in D} a_i p_i' = \sum_{t \in \mathbf{N}} \sum_{i \in D_t} a_i p_i' > \sum_{t > 0} 1 = \infty$, whereas $|z|_\pi = \sum_{i \in D} a_i p_i = \sum_{t \in \mathbf{N}} \sum_{i \in D_t} a_i p_i \leqq \sum_{t > 0} (M' + 1) 2^{-t} < \infty$. So $z \in S_\pi \backslash S_{\pi'}$ and we have arrived at a contradiction. So, from $S_{\pi'}(M') \supset S_\pi(M)$, we have concluded that, for some $E > 1$, $\sum_{i \in F_E} k_i p_i'$ is finite and hence $\sum_{i \in F_E} k_i p_i$ is also finite.

We define $r_i = p_i'$ for $i \in F_E$ and $r_i = p_i$ for $i \in V_\pi(M) \backslash F_E$.

We obtain: $p_i'/r_i < E$ for all $i \in V_\pi(M)$ and

$$\sum_{i \in V(M)} k_i |p_i - r_i| = \sum_{i \in F_E} k_i |p_i - p_i'| < \sum_{i \in F_E} (k_i p_i + k_i p_i') < \infty.$$

We summarize what we have obtained in

68. Lemma. $S_{\pi'} \supset S_\pi$ if and only if there exists a set $V \subset \mathbf{N}$ and a set $\{r_i \in \mathbf{R}^+ | i \in V\}$ such that

(i) $\bigcup_{i \in V} \{p_i, p_i'\}$ is bounded,

(ii) $\{p_i | i \notin V\}$ has a nonzero lower bound and

(iii) $\{p_i'/r_i | i \in V\}$ has an upperbound and $\sum_{i \in V} k_i |p_i - r_i| < \infty$.

Proof. Choose M' arbitrary, and choose M such that the conditions of 67. are fulfilled. Put $V = V_\pi(M)$.

69. Lemma. $S_{\pi'} = S_\pi$ if and only if there exists a set $V \subset \mathbf{N}$ and a set $\{r_i \in \mathbf{R}^+ | i \in V\}$ such that

(i) $\bigcup_{i \in V} \{p_i, p_i'\}$ is bounded,

(ii) $\bigcup_{i \notin V} \{p_i, p_i'\}$ has a non-zero lower bound and

(iii) $\{p_i'/r_i | i \in V\}$ has an upperbound and a non-zero lowerbound, and $\sum_{i \in V} k_i |p_i - r_i| < \infty$.

Proof. "If". Suppose V and $\{r_i | i \in V\}$ given. Then, for $x = \sum a_i n_i \in G_\vartheta$, $x \in S_\pi$ if and only if $\sum_{i \in \mathbf{N}} a_i p_i < \infty$ if and only if $\sum_{i \in V} a_i p_i < \infty$ and $\{i \notin V | a_i \notin V\}$ is finite. $\sum_{i \in V} a_i p_i < \infty$ if and only if $\sum_{i \in V} k_i |p_i - r_i| < \infty$ if and only if $\sum_{i \in V} a_i p_i'$ is finite. So $x = \sum_{i=0}^\infty a_i n_i \in S_\pi$ if and only if $\sum_{i \in V} a_i p_i' < \infty$ and $\{i \notin V | a_i \neq 0\}$ is finite and this is equivalent with $x = \sum a_i n_i \in S_{\pi'}$.

"Only if". Suppose $S_\pi = S_{\pi'}$. Then, by 67., for every M' there exist M and M'' such that $M' > M > M'' > 0$ and such that $V_{\pi'}(M') \supset V_\pi(M) \supset \supset V_{\pi'}(M'')$. Put $V = V_\pi(M)$. Then, on V, the p_i are bounded by M and the p_i' are bounded by M'. On $\mathbf{N} \backslash V$, the p_i have M as lower bound and the p_i' have M'' as lower bound.

Suppose that p_i'/p_i is unbounded on V, then, as in the second part of 67., we find that, for E great enough, $\sum_{i\in F_E} k_i p_i' < \infty$ and $\sum_{i\in F_E} k_i p_i < \infty$ where $F_E = \{i \in V | p_i'/p_i > E\}$. Likewise we find that $k_i p_i$ and $k_i p_i'$ are summable on $f_e = \{i \in V | p_i'/p_i < 1/e\}$, for e large enough. Choose $r_i = p_i$ for $i \notin F_E \cup f_e$, and $r_i = p_i'$ for $i \in F_E \cup f_e$. Then clearly $1/e < p_i'/r_i < E$ and $\sum_{i\in V} k_i|p_i - r_i| = \sum_{i\in F_E \cup f_e} k_i|p_i - p_i'| < \infty$. This finishes the proof.

§ 8. An application: A complete nondiscrete group-topology on Z

In the following $\vartheta = \{(i+1)! | i \in \mathbf{N}\}$ and, for $A \subset \mathbf{N}$, $p_i = \frac{1}{2}$ if $i \in A$, $p_i = 1/(i+2)$ if $i \notin A$ and $\varrho_A = \{p_i | i \in \mathbf{N}\}$. $\pi_A = \{\vartheta, \varrho_A\}$ defines a DS-topology, because $(i+2)!p_i > (i+1)!p_{i+1}$. In fact, $1 = (i+2)p_i > \frac{1}{2} > 1/(i+3)$, for $i \in \mathbf{N}$. We write $\|z\|_A$ instead of $\|z\|_{\pi_A}$. $\langle \mathbf{Z}, \pi_A \rangle$ is nondiscrete if and only if A has an infinite complement in $\mathbf{N}$. If S is any set we will denote the set of all subsets of S by $\mathrm{Pow}(S)$, the powerset of S.

We recall that an ideal on S is a subset $\mathscr{F}$ of $\mathrm{Pow}\,(S)$ with the property: for all $A \in \mathscr{F}$ and $B \in \mathrm{Pow}\,(S)$, $A \cap B \in \mathscr{F}$ and for all $A \in \mathscr{F}$ and $B \in \mathscr{F}$, $A \cup B \in \mathscr{F}$. We remark that, if $A \in \mathrm{Pow}\,(\mathbf{N})$, $B \in \mathrm{Pow}\,(\mathbf{N})$ and $A \supset B$, then $1_{\mathbf{Z}}: \langle \mathbf{Z}, \pi_A \rangle \to \langle \mathbf{Z}, \pi_B \rangle$ is continuous.

70. **Lemma.** Suppose $\mathscr{F}$ is an ideal of subsets of $\mathbf{N}$ that have an infinite complement in $\mathbf{N}$. Then the coarsest topology on $\mathbf{Z}$, finer than every π_A-topology with $A \in \mathscr{F}$, is non-discrete.

Proof. We call this topology $T(\mathscr{F})$. A basis of neighborhoods for $T(\mathscr{F})$ is $\bigcup_{A\in\mathscr{F}} \mathfrak{N}_A$, where $\mathfrak{N}_A$ stands for a basis of neighborhoods at 0 in $\langle \mathbf{Z}, \pi_A \rangle$. Because none of the $\langle \mathbf{Z}, \pi_A \rangle$, with $A \in \mathscr{F}$, is discrete, $T(\mathscr{F})$ is non-discrete.

71. **Lemma.** Suppose $\mathscr{F}$ is an ideal of sets with infinite complement on $\mathbf{N}$. Suppose for every infinite subset B of $\mathbf{N}$, there exists an infinite $A \in \mathscr{F}$ with $A \subset B$. Then $\langle \mathbf{Z}, T(\mathscr{F}) \rangle$ is complete.

Proof. A net in $\mathbf{Z}^+$ is a $T(\mathscr{F})$-Cauchy net, if and only if it is a Cauchy net with respect to every π_A-topology, with $A \in \mathscr{F}$. Suppose it does not converge to an element of $\mathbf{Z}$. Then, in G_ϑ it will converge to some $z \in G_\vartheta \setminus \mathbf{Z}$, say $z = \sum_{i=0}^{\infty} a_i(i+1)!$, in which infinitely many $a_i \neq 0$ such that, for no i_0 does $i > i_0$ imply $a_i = i+1 = k_i - 1$. Now when $a_i \notin \{0, i+1\}$, for $i \in \mathbf{N}$, the i-th place contributes at least $p_i \neq 0$ to $\|z\|_A$, whatever A is. We use here the terminology introduced in § 5, immediately preceding 20. Now let $x \in \mathbf{Z}^+$, $x = \sum_{i=0}^{\infty} b_i(i+1)!$. Suppose that $b_i = i+1 = k_i - 1$ and $b_{i+1} = 0$. Then the i-th and the $i+1$-st place contribute at least p_{i+1}.

We can see this as follows. Let y be minimizing for x. Either $Q(i+1, x, y) = 1$, or $Q(i+1, x, y) = 0$. In the first case, there is a contribution of at least p_{i+1} from the $i+1$-st place to the norm of x. In the second case, it follows that there is a contribution of

$$(k_i - 1)p_i \geqq (i+1)/(i+2) \geqq \max\,(\tfrac{1}{2}, 1/(i+3)) \geqq p_{i+1}$$

from the i-th place. Let us consider $B=\{i \in \mathbf{N} | a_i \notin \{0, i+1\}$ or $(a_{i-1}=i$ and $a_i=0)\}$. Because $z \notin \mathbf{Z}$, B is infinite. Hence there exists an infinite $A \in \mathscr{F}$ with $A \subset B$. We have assumed $\|z\|_A<\infty$. But any net $\{z_\lambda | \lambda \in \Lambda\}$, for some directed set Λ, that converges coordinatewise to z (cf. 35), satisfies $\lim \{\|z_\lambda\|_A | \lambda \in \Lambda\}=\infty$. We have arrived at a contradiction.

72. Definition. Let $\langle M, S \rangle$ be a pair consisting of a topological space S and a dense subset M of S. A proper ideal I on S is called *clustering over* M if it has the property that for every infinite set $X \subset M$, there is an infinite $Y \subset X$, such that $\overline{Y}$ (the closure of Y in S) is an element of I.

73. Definition. Let S_1 and S_2 be sets and $\varphi: S_1 \to S_2$ a map. For an ideal I on S_2, we let $\varphi^{-1}I$ denote the ideal consisting of all $X \subset S_1$, such that $X \subset \varphi^{-1}X'$ for some $X' \in I$.

74. Lemma. Let M_1 be dense in S_1 and M_2 dense in S_2. Suppose $\varphi: S_1 \to S_2$ is a continuous map which is surjective and which has the property that $\varphi M_1 \backslash M_1$ is finite and $M_1 \cap \varphi^{-1}\{\varphi x\}$ is finite for all $x \in M_1$. If I is an ideal on S_2, clustering over M_2, then $\varphi^{-1}I$ is clustering over M_1.

Proof. Because φ is onto, I is proper only if $\varphi^{-1}I$ is proper. Let X be an infinite subset of M_1. φX contains an infinite subset of M_2, hence there is a $Y \subset M_2 \cap \varphi X$ such that $\overline{Y} \in I$. Hence $\varphi^{-1}\overline{Y}$ is closed in S_1 and $\varphi^{-1}Y$ contains an infinite set contained in M_1, say Z_1. $Z_1 \subset \varphi^{-1}\overline{Y}$, hence $\overline{Z}_1 \subset \varphi^{-1}\overline{Y}$, so $\overline{Z}_1 \in \varphi^{-1}I$, which proves the lemma.

75. Definition. The Stone-Čech compactification of $\mathbf{N}$ is denoted by $\mathbf{BN}$. β denotes the natural injection $\mathbf{N} \to \mathbf{BN}$. We write $\mathrm{cl}\,(X)$ for the closure of $\beta(X)$ in $\mathbf{BN}$, for any subset X of $\mathbf{N}$.

76. Lemma. $\mathrm{cl}: X \to \mathrm{cl}\,(X)$ is a homomorphism of the Boolean algebra Pow $(\mathbf{N})$ into the Boolean algebra Pow $(\mathbf{BN})$.

Proof. Evident.

77. Lemma. Let $\mathscr{F}$ be an ideal on $\mathbf{N}$. Let $\mathrm{cl}\,(\mathscr{F})$ be the ideal on $\mathbf{BN}$ consisting of all $X \subset \mathbf{BN}$, for which there is a $Y \in \mathscr{F}$, such that $X \subset \mathrm{cl}\,(Y)$. Then the condition that every $A \in \mathscr{F}$ has an infinite complement and that for every infinite $B \subset \mathbf{N}$, there exists an infinite $A \subset B$, such that $A \in \mathscr{F}$, is equivalent with the statement: $\mathrm{cl}\,(\mathscr{F})$ is an ideal on $\mathbf{BN}$, clustering over $\beta(\mathbf{N})$.

Proof. $\beta^{-1}(\mathrm{cl}\,(\mathscr{F}))=\mathscr{F}$, for let $X' \in \mathscr{F}$ and Y be such that $X \subset \beta^{-1}Y$, and $Y \subset \mathrm{cl}\,(X')=\overline{\beta(X')}$, then $\beta(X) \subset \overline{\beta(X')}$, hence $X \subset X'$, hence $X \in \mathscr{F}$. Every $A \in \mathscr{F}$ has infinite complement is equivalent with $\mathrm{cl}\,(A) \neq \mathbf{BN}$ for all $A \in \mathscr{F}$, hence equivalent with the statement: $\mathrm{cl}\,(\mathscr{F})$ is proper. The rest is trivial.

78. Example 1. An ideal on **BN**, clustering over $\beta(\mathbf{N})$. The ideal of all sets not containing $n \in \mathbf{BN}\backslash\beta(\mathbf{N})$.

Proof. Let $X \subset \mathbf{N}$ be infinite. X contains two disjoint infinite sets I, J. $\mathrm{cl}\,(I) \cap \mathrm{cl}\,(J) = \emptyset$, hence they cannot both contain n. So $\mathrm{cl}\,(I)$ or $\mathrm{cl}\,(J)$ is in the ideal.

79. Example 2. An ideal on $K(\mathbf{N})$ clustering over $k(\mathbf{N})$, in which $k: \mathbf{N} \to K(\mathbf{N})$ is an injection of **N** into some compact metrizable space $K(\mathbf{N})$, such that $k(\mathbf{N})$ is dense in $K(\mathbf{N})$.

The set of all closed subsets S of $K(\mathbf{N})$ with the property that the cardinality of the set of nonisolated points of S is less than some infinite cardinal number $\mathfrak{m}$, less than or equal to the cardinality of $K(\mathbf{N})$.

Comment. In the case of example 2, there is a continuous map $\varphi: \mathbf{BN} \to K(\mathbf{N})$, such that $\varphi\beta = k$. Let I be a clustering ideal defined as in the example, then $k^{-1}I = \beta^{-1}(\varphi^{-1}I)$ and we can think of example 2 as a way of obtaining ideals in **BN**, clustering over $\beta(\mathbf{N})$.

We have arrived at a wealth of ways to produce non-discrete complete topologies on **Z**. Actually we have:

80. Lemma. The cardinality of the set of non-discrete complete group-topologies on **Z** equals the cardinality of the set of all group-topologies on **Z**, namely the cardinality of Pow (Pow (**N**)).

Proof. The ideals $\beta^{-1}I$ on **N**, which we obtain by example 1 are just the maximal ideals on **N**. We prove that for two different maximal ideals $\mathscr{F}$ and $\mathscr{F}'$, the resulting topologies on **Z** are different. As the cardinality of the set of all maximal ideals equals the cardinality of the set of all ultrafilters and as this cardinality equals the cardinality of Pow (Pow (**N**)), we will then have proved the result.

Let $\mathscr{F}$ and $\mathscr{F}'$ be two different maximal ideals. Consider the directed set Λ consisting of sets $\lambda = \lambda(A, n) = \{i \in \mathbf{N} | i \notin A \text{ and } i \geq n\}$ for $A \in \mathscr{F}$. Λ is ordered by $\lambda' > \lambda$ if and only if $\lambda' \subset \lambda$. We define the net $\{z_\lambda | \lambda \in \Lambda\}$ by $z_\lambda = (k+1)!$, where $k = \min(\lambda)$. For $A \in \mathscr{F}$, for all $\lambda = \lambda(B, n) \subset \lambda(A, 0)$ holds that $\|z_\lambda\|_B \leq 1/(n+2)$.

Now if $\mathscr{F}$ and $\mathscr{F}'$ are different maximal ideals take $A \in \mathscr{F}\backslash\mathscr{F}'$. Because $A \notin \mathscr{F}'$ and $\mathscr{F}'$ is maximal, $\mathbf{N}\backslash A = C \in \mathscr{F}'$. So, for all $\lambda = \lambda(B, n) \subset \subset \lambda(A, 0)$, $\|z_\lambda\|_C = \frac{1}{2}$. So $\{z_\lambda\}$ is a Cauchy net converging to 0 in the topology defined by $\mathscr{F}$ but not in the topology defined by $\mathscr{F}'$. We have proved the lemma.

We combine 14., 71., 78. and the proof of 80. in:

81. Lemma. Let $\mathscr{F}$ be a maximal ideal on **N**, containing all finite subsets of **N**.

Define

$$V_{A,\varepsilon}=\{z=\sum_{i=0}^{\infty} a_i(i+1)! \in \mathbf{Z}^+|(a_i \in \mathbf{Z}^+,\ 0 \leqq a_i < i+1) \text{ and } (i \in A \text{ implies } a_i=0) \text{ and } (\sum_{i=0}^{\infty} a_i/(i+2)<\varepsilon)\},$$

for $A \in \mathscr{F}$ and $\varepsilon < \frac{1}{2}$.

Then $\{V_{A,\varepsilon}-V_{A,\varepsilon}|A \in \mathscr{F}$ and $0<\varepsilon<\frac{1}{2}\}$ is a basis of neighborhoods for a topology $T(\mathscr{F})$ on $\mathbf{Z}$, such that $\langle \mathbf{Z}, T(\mathscr{F}) \rangle$ is a nondiscrete, complete, Hausdorff topological group. Moreover if $\mathscr{F} \neq \mathscr{F}'$, then $\langle \mathbf{Z}, T(\mathscr{F}) \rangle \neq \langle \mathbf{Z}, T(\mathscr{F}') \rangle$.

Proof. Clear from the preceding.

§ 9. A CLASS OF NORMED GROUPS HOMEOMORPHIC TO COMPLETE DS-GROUPS

In this section P will stand for $\prod\{\mathbf{R}/\mathbf{Z}|i \in \mathbf{N}\}$, the infinite dimensional torus with the usual topology of coordinatewise convergence. S will stand for $\sum\{\mathbf{R}/\mathbf{Z}|i \in \mathbf{N}\} \subset P$. An element of P will be denoted by $\{x_i\}$; so S consists of all $\{x_i\} \in P$ such that $x_i \neq 0$ for only a finite number of indices i. Let $T_n=\{\{x_i\} \in S|x_i=0$ for $i>n\}$ and $S_n=\{\{x_i\} \in S|x_i=0$ for $i \leqq n\}$. So $S=S_n+T_n$.

Let m be some group norm on S, satisfying that the restriction of m to T_n induces the usual compact topology on T_n. Let M be the completion of S for m and let $\varphi\colon M \to P$ be the canonical morphism, then we have

82. Lemma. Suppose φ is injective. Then M is compact if and only if φ is surjective.

Proof. "If" is trivial, so let φ be injective and M compact. φM is also compact. Let $q\colon P \to P/\varphi M$ be the quotient morphism. $P/\varphi M$ is compact so, for a character χ of $P/\varphi M$, χq is a character of P vanishing on M, hence on S, hence $\chi=0$, so $P/\varphi M=0$, and $\varphi M=P$.

83. Lemma. If S_n is closed in S with respect to the m-topology then M is compact if and only if φ is an isomorphism.

Proof. $M=T_n+M_n$, where M_n is the completion of S_n relative to m (or the closure of S_n in M). If M is compact, then

$$M=\operatorname{lim\,proj} M/M_n=\operatorname{lim\,proj} T_n=P.$$

Suppose now that φ is injective and $M \neq P$. An element x of M, that generates a dense subgroup of P, has the property that $\overline{x\cdot\mathbf{Z}}$ (closure taken in M) is not compact. If $\overline{x\cdot\mathbf{Z}}$ is not discrete either, we have an example of a monothetic group which is not locally compact. In fact, S. Rolewicz [5] exhibited a case in which $\overline{x\cdot\mathbf{Z}}=M\neq P$. Even when $\overline{x\cdot\mathbf{Z}}\neq M$, $\overline{x\cdot\mathbf{Z}}$ may provide an example of a not locally compact monothetic group.

84. Definitions. $|\star|_a$ on $\mathbf{R}$ denotes the usual absolute value norm,

$$|\star| \text{ on } \mathbf{R}/\mathbf{Z} \text{ is defined by } |x+\mathbf{Z}| = \min_{n\in\mathbf{Z}} |x+n|_a$$

and

$$|\star| \text{ on } \mathbf{Z}/(t) \text{ is defined by } |x+t\mathbf{Z}| = \min_{n\in\mathbf{Z}} |x+tn|_a.$$

Let $\mu=\{q_i|i\in\mathbf{N}\}$ be a sequence of positive reals, then m_μ, defined by $m_\mu(\{x_i\}) = \sum_{i=0}^{\infty} |x_i|q_i$, is a norm on S satisfying T_n is compact and S_n is closed.

85. Definition. For $u=\langle \vartheta, \mu \rangle$, in which $\vartheta=\{n_i\in\mathbf{Z}^+|i\in\mathbf{N}\}$, $n_0=1$, $n_i\neq n_{i+1}$, $n_i|n_{i+1}$ for all $i\in\mathbf{N}$ and in which $\mu=\{q_i\in\mathbf{R}^+\backslash\{0\} \mid i\in\mathbf{N}\}$ such that $\sum_{i=0}^{\infty} q_i/(n_{i+1})<\infty$, define $\|\star\|_u$ on $\mathbf{Z}$ by $\|z\|_u=m_\mu(\{z/(n_{i+1}) \bmod 1\})$.

86. Definition. For $u=\langle \vartheta, \mu \rangle$, satisfying the conditions of 85., define $\pi=\{\vartheta, s\mu\}$ by $s\mu=\{\|n_i\|_u|i\in\mathbf{N}\}$.

For $\pi=\{\vartheta, \varrho\}$ define $u=\langle \vartheta, d\varrho \rangle$ by $d\varrho=\{k_ip_i-p_{i+1}|i\in\mathbf{N}\}$. In this, $k_i=n_{i+1}/n_i$, as before.

87. Lemma. $\langle \vartheta, d\varrho \rangle$ satisfies the conditions of 85. if and only if $d\varrho$ consists of non-zero real numbers. $\{\vartheta, s\mu\}$ defines a DS-groupnorm, and moreover: $k_ip_i>p_{i+1}$ for all i.

Proof. If $d\varrho$ consists of non-zero numbers, then

$$\sum_{i=0}^{t} (k_ip_i-p_{i+1})/n_{i+1} = \sum_{i=0}^{t} p_i/n_i - \sum_{i=0}^{t} p_{i+1}/n_{i+1} = p_0 - p_{t+1}/n_{t+1}$$

as p_{t+1}/n_{t+1} is a decreasing positive function of t, we have proved the first statement. Secondly, for $\varrho=s\mu$,

$$p_j=\|n_j\|_u=\sum_{i=j}^{\infty} n_j/n_{i+1}q_i. \text{ So } (n_{j+1}/n_j)\|n_j\|_u-\|n_{j+1}\|_u=$$
$$=\sum_{i=j}^{\infty} (n_{j+1}/n_{i+1})q_i-\sum_{i=j+1}^{\infty} (n_{j+1}/n_{i+1})q_i=q_j>0.$$

88. Lemma. $\langle \vartheta, \mu \rangle=\langle \vartheta, ds\mu \rangle$, $\{\vartheta, sd\varrho\}=\{\vartheta, \varrho\}$ if and only if $\lim_{i\in\mathbf{N}} p_i/n_i=0$.

Proof. The first statement follows from the proof of 87. Putting $q_i=k_ip_i-p_{i+1}$ for all i, we find

$$\|n_j\|_u=\sum_{i=j}^{\infty} (n_j/n_{i+1})(k_ip_i-p_{i+1})=n_j\sum_{i=j}^{\infty} (p_i/n_i-p_{i+1}/n_{i+1})=n_j(p_j/n_j-l)$$

in which $l=\lim_{i\in\mathbf{N}} p_i/n_i$, so we are done.

Let $u=\langle \vartheta, \mu \rangle$ be given. Then we will prove that $\|\star\|_u$ and $\|\star\|_\pi$ are equivalent, with $\pi=\{\vartheta, s\mu\}$. First we need a few technical lemmas.

89. Lemma. Let $\vartheta=\{n_i\}$ be given. For every $z\in\mathbf{Z}$, there exists a decomposition $z=\sum_{i=0}^{\infty} b_in_i$, with $b_i\in\mathbf{Z}$ for all i, and such that only a finite number of b_i are unequal to zero, $|b_i|_a\leq\frac{1}{2}k_i$ for all i, and such that if $|b_i|_a=\frac{1}{2}k_i$, then not $|b_{i+1}|_a=\frac{1}{2}k_{i+1}$ and not sign $b_i=-$ sign b_{i+1}.

Proof. We define b_i by induction. $z \in a_0 + n_1\mathbf{Z}$. Choose a representative b_0 of $a_0 + n_1\mathbf{Z}$, such that $|b_0|_a \leqq \frac{1}{2}n_1 = \frac{1}{2}k_1$.

Suppose b_i already defined for $0 \leqq i \leqq j$. Then $z = \sum_{i=0}^{j} b_i n_i + n_{j+1}y$. $y \in a_{j+1} + k_{j+1}\mathbf{Z}$. Choose a representative b_{j+1} from $a_{j+1} + k_{j+1}\mathbf{Z}$, such that $|b_{j+1}|_a \leqq \frac{1}{2}k_{j+1}$. Suppose $|b_j|_a = \frac{1}{2}k_j$ and $|b_{j+1}|_a = \frac{1}{2}k_{j+1}$, that is for instance

$$z = \sum_{i=0}^{j-1} b_i n_i + k_j n_j/2 - k_{j+1}n_{j+1}/2 + y.$$

Then $z = \sum_{i=0}^{j-1} b_i n_i - k_j n_j/2 - (k_{j+1}/2 - 1)n_{j+1} + y$, and we see that, by redefining b_j, we can put things straight again. The rest is similar.

90. Lemma. Let $z = \sum_{i=0}^{\infty} b_i n_i$, with the b_i chosen such that they satisfy the conditions of 89. Then $\sum_{i=0}^{j} |b_i|_a n_i < \frac{3}{5}n_{j+1}$ if $|b_j|_a < \frac{1}{2}k_j$, and $\sum_{i=0}^{j} |b_i|_a n_i < \frac{4}{5}n_{j+1}$ if $|b_j|_a = \frac{1}{2}k_j$.

Proof. The statement is evident for $j=0$. Now suppose it has been proved for $0 \leqq j \leqq m-1$. Suppose $|b_m|_a < \frac{1}{2}k_m$; so by induction

$$\sum_{i=0}^{m-1} |b_i|_a n_i < n_m(\tfrac{4}{5} + |b_m|_a) \leqq (\tfrac{4}{5} + \tfrac{1}{2}k_m - \tfrac{1}{2})n_m \leqq \tfrac{3}{5}k_m n_m,$$

because we may assume $k_m \neq 2$ (in the case $k_m = 2$, $b_m = 0$, so $\frac{4}{5} < \frac{3}{5}k_m$).

If $|b_m|_a = \frac{1}{2}k_m$ then $b_{m-1} < \frac{1}{2}k_{m-1}$ so the induction hypothesis shows that $\sum_{i=0}^{m} |b_i|_a n_i < (\frac{3}{5} + \frac{1}{2}k_m)n_m \leqq \frac{4}{5}k_m n_m$.

91. Lemma. Suppose $z = \sum_{i=0}^{\infty} b_i n_i$ as in 89.
Then $|\sum_{i=0}^{j} b_i n_i|_a > \frac{1}{4}\sum_{i=0}^{j} |b_i|_a n_i$, for all j.

Proof. Suppose proved for $0 \leqq j \leqq m-1$ (evident for $j=0$). If $b_m = 0$ then trivial for $j=m$. Otherwise put $F = |\sum_{i=0}^{m} b_i n_i|_a$, $G = \sum_{i=0}^{m} |b_i|_a n_i$. Suppose first $|b_m|_a > 1$. Then $F \geqq |b_m|_a n_m - \sum_{i=0}^{m-1} |b_i|_a n_i > \frac{3}{7}G$, by 90. If $|b_m|_a = 1$ and $|b_{m-1}|_a < \frac{1}{2}k_{m-1}$, then we get $F > \frac{1}{4}G$. If $b_m = 1$ and $b_{m-1} = \frac{1}{2}k_{m-1}$, $F \geqq \frac{3}{2}n_m - \sum_{i=0}^{m-2} |b_i|_a n_i > \frac{3}{7}G$, by 90.

92. Lemma. Let $u = \langle \vartheta, \mu \rangle = \langle \{n_i\}, \{q_i\} \rangle$ and let $\pi = \{\vartheta, s\mu\} = \{\{n_i\}, \{\|n_i\|_u\}\}$. Then $\sum_{j=0}^{\infty} 1/(n_{j+1})q_j \sum_{i=0}^{j} a_i n_i = \sum_{j=0}^{\infty} p_j a_j$ for all sets $\{a_i \in R \mid i \in \mathbf{N}\}$, such that both sides are absolutely convergent.

Proof. By the proof of 87., $q_j = k_j p_j - p_{j+1}$. So

$$\begin{aligned}\sum_{j=0}^{\infty} (1/n_{j+1})q_j \sum_{i=0}^{j} a_i n_i &= \sum_{j=0}^{\infty} (p_j/n_j - p_{j+1}/n_{j+1}) \sum_{i=0}^{j} a_i n_i = \\ &= \sum_{j=0}^{\infty} (p_j/n_j) \sum_{i=0}^{j} a_i n_i - \sum_{j=0}^{\infty} (p_{j+1}/n_{j+1}) \sum_{i=0}^{j+1} a_i n_i + \\ &+ \sum_{j=0}^{\infty} (p_{j+1}/n_{j+1}) a_{j+1} n_{j+1} = \sum_{j=0}^{\infty} p_j a_j.\end{aligned}$$

93. Theorem. For all $x \in \mathbf{Z}$, $\|x\|_\pi \geqq \|x\|_u > \frac{1}{16}\|x\|_\pi$.

Proof. Clearly $\|x\|_\pi \geqq \|x\|_u$, because $\|\star\|$ is the largest norm such that $\|n_j\| = \|n_j\|_u$. For $x \in \mathbf{Z}$, write $x = \sum_{i=0}^{\infty} b_i n_i$ as in 89. Then

$$|(\textstyle\sum_{i=0}^{j} b_i n_i) \bmod n_{j+1}| > \frac{1}{4} |\textstyle\sum_{i=0}^{j} b_i n_i|$$

by 90.

Then 91. gives $|(\sum_{i=0}^{j} b_i n_i) \bmod n_{j+1}| > \frac{1}{16} \sum_{i=0}^{j} |b_i|_a n_i$. Hence 92. gives $\|x\|_u > \frac{1}{16} \|x\|_\pi$.

We have proved what we announced, namely that $\|\star\|_u$ and $\|\star\|_\pi$ are equivalent.

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